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Exposed 2-Homogeneous Polynomials on the two-Dimensional Real Predual of Lorentz Sequence Space

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Abstract. We classify the exposed polynomials of the unit ball of the space of 2-homogeneous polynomials on the two-dimensional real predual of Lorentz sequence space. In fact, we prove that

$$\text{exp}B_{\mathcal{P}(^2d_*(1,w)^2)} = \text{ext}B_{\mathcal{P}(^2d_*(1,w)^2)} \Big\backslash \left\{ \pm\left[\frac{x^2 - y^2 \pm 2wxy}{1+w^2}\right], \right.$$
$$\left. \pm\left[\frac{1-w}{(1+w)(1+w^2)}(x^2-y^2) \pm \frac{2}{(1+w)^2}xy\right]\right\}.$$

Mathematics Subject Classification. Primary 46A22.

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1. Introduction

We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \backslash \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\text{ext}B_E$ and $\text{ext}B_E$ the sets of exposed and extreme points of B_E, respectively. A mapping $P\colon E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous symmetric bilinear form L on the product $E \times E$ such that $P(x) = L(x,x)$ for every $x \in E$. We denote by $\mathcal{L}_s(^2E)$ the Banach space of all continuous symmetric bilinear forms on E endowed with the norm $\|L\| = \sup_{\|x\|=\|y\|=1} |L(x,y)|$. $\mathcal{P}(^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. For more details about

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the theory of polynomials on a Banach space, we refer to [7]. In 2003, Kim and Lee [23] studied exposed 2-homogeneous polynomials on Hilbert spaces. Later, Choi and Kim [6] characterized the exposed points of the unit ball of $\mathcal{P}(^2l_p^2)$ $(p = 1, 2, \infty)$ and in 2007, Kim [15] characterized the exposed points of the unit ball of $\mathcal{P}(^2l_p^2)$ $(1 < p < \infty, p \neq 2)$. We refer to ([1–6,8–31] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. Let $0 < w < 1$ be fixed. We denote the two-dimensional real predual of Lorentz sequence space by

$$d_*(1,w)^2 := \left\{(x,y) \in \mathbb{R}^2 \colon \|(x,y)\|_{d_*} := \max\left\{|x|, |y|, \frac{|x|+|y|}{1+w}\right\}\right\}.$$

In fact, the two-dimensional real predual of Lorentz sequence space $d_*(1,w)^2$ is the plane $\mathbb{R}^2$ with the octagonal norm of weight w. We will denote by $P(x,y) = ax^2 + by^2 + cxy$ a 2-homogeneous polynomial on $d_*(1,w)^2$. In 2011, Kim [17] computed the norm of $P \in \mathcal{P}(^2d_*(1,w)^2)$ in terms of its real coefficients and determined all the extreme polynomials of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$. Recently, Kim [19] classified all the smooth polynomials of the unit ball of $\mathcal{P}(^2d_*(1,w)^2)$. In this paper, using results of the previous works [17,19,22], we classify the exposed polynomials of the unit ball of the space $\mathcal{P}(^2d_*(1,w)^2)$. Indeed, we will show that

$$\begin{aligned}\mathrm{exp}B_{\mathcal{P}(^2d_*(1,w)^2)} = \mathrm{ext}B_{\mathcal{P}(^2d_*(1,w)^2)} \Big\backslash \Big\{ &\pm\left[\frac{x^2-y^2\pm 2wxy}{1+w^2}\right], \\ &\pm\left[\frac{1-w}{(1+w)(1+w^2)}(x^2-y^2) \pm \frac{2}{(1+w)^2}xy\right]\Big\}.\end{aligned}$$

2. The Results

Let $P \in \mathcal{P}(^2d_*(1,w)^2)$ with $P(x,y) = ax^2 + by^2 + cxy$ for $(x,y) \in d_*(1,w)^2$. Note that if $\|P\| = 1$, then $|a| \leq 1, |b| \leq 1$ and $|c| \leq \frac{4}{(1+w)^2}$. Indeed, $\|P\| \geq |P(\pm(\frac{1+w}{2}),\ \frac{1+w}{2})| = \frac{(1+w)^2}{4}|a+b\pm c| = \frac{(1+w)^2}{4}(|a+b|+|c|) \geq \frac{(1+w)^2}{4}|c|$. Since

$$\|ax^2 + by^2 + cxy\| = \|bx^2 + ay^2 \pm cxy\| = \|-bx^2 - ay^2 \pm cxy\|,$$

we may assume that $a \geq |b| \geq 0,\ c \geq 0$.

Theorem 1 [17, Theorem 1]. *Let $P \in \mathcal{P}(^2d_*(1,w)^2)$ with $P(x,y) = ax^2 + by^2 + cxy$ for $(x,y) \in d_*(1,w)^2$ with $a \geq |b| \geq 0,\ c \geq 0$. Then,*

Case 1: $0 \leq c < 2|b|$

Subcase 1: $b < 0$

(a) *If* $\frac{c}{2|b|} \leq w$, *then*

$$\|P\| = a + \frac{c^2}{4|b|}.$$

(b) *If* $\frac{c}{2|b|} > w$, *then*

$$\|P\| = bw^2 + cw + a.$$

Subcase 2: If $b > 0$, *then*

$$\|P\| = bw^2 + cw + a.$$

Case 2: If $2|b| \le c \le 2a$, *then*

$$\|P\| = bw^2 + cw + a.$$

Case 3: $2a < c$

(a) *If* $\frac{c-2a}{c-2b} < w$, *then*

$$\|P\| = bw^2 + cw + a.$$

(b) *If* $\frac{c-2a}{c-2b} \ge w$, *then*

$$\|P\| = \frac{(c^2 - 4ab)(1+w)^2}{4(c-a-b)}.$$

Theorem 2 [17, Theorem 2].

$$\begin{aligned} \mathrm{ext}B_{\mathcal{P}(^2d_*(1,w)^2)} = \Bigg\{ & \pm x^2,\ \pm y^2,\ \pm \frac{1}{1+w^2}(x^2+y^2),\ \pm \frac{1}{(1+w)^2}(x \pm y)^2 \\ & \pm\Big[t(x^2-y^2) \pm 2\sqrt{t(1-t)}xy\Big] \ \left(\frac{1}{1+w^2} \le t \le 1\right), \\ & \pm \left[t(x^2-y^2) \pm \frac{2+2\sqrt{1-t^2(1+w)^4}}{(1+w)^2}xy\right] \\ & \left(0 \le t \le \frac{1-w}{(1+w)(1+w^2)}\right)\Bigg\}. \end{aligned}$$

Theorem 3 [19, Theorem 3]. *Let* $f \in \mathcal{P}(^2d_*(1,w)^2)^*$ *and* $\alpha = f(x^2), \beta = f(y^2), \gamma = f(xy)$. *Then,*

$$\begin{aligned} \|f\| = \max\Bigg\{ & |\alpha|,\ |\beta|,\ \frac{1}{1+w^2}|\alpha+\beta|,\ \frac{1}{(1+w)^2}(|\alpha+\beta| + 2|\gamma|), \\ & t|\alpha-\beta| + 2\sqrt{t(1-t)}|\gamma| \ \left(\frac{1}{1+w^2} \le t \le 1\right), \\ & t|\alpha-\beta| + \frac{2+2\sqrt{1-t^2(1+w)^4}}{(1+w)^2}|\gamma| \ \left(0 \le t \le \frac{1-w}{(1+w)(1+w^2)}\right)\Bigg\}. \end{aligned}$$

Observe that if $0 < w < 1$ and $w^* = \frac{1-w}{1+w}$, then $0 < w^* < 1$ and $(w^*)^* = w$.

Lemma 4 [22, Lemma 2.4]. *Let* $w^* = \frac{1-w}{1+w}$. *Then, there is an isometry* $\phi\colon d_*(1,w) \to d_*(1,w^*)$ *such that*

$$\phi(x,y) := \left(\frac{x+y}{1+w}, \frac{x-y}{1+w}\right).$$

Proof. By definition, the norms of $(x,y) \in d_*(1,w)$ and $(X,Y) \in d_*(1,w^*)$ are given by

$$\|(x,y)\|_{d_*(1,w)} = \max\left\{|x|,|y|,\frac{|x|+|y|}{1+w}\right\},$$
$$\|(X,Y)\|_{d_*(1,w^*)} = \max\left\{|X|,|Y|,\frac{|X|+|Y|}{1+w^*}\right\}.$$

Now, let $(X,Y)=\phi(x,y)=\left(\frac{x+y}{1+w},\frac{x-y}{1+w}\right)$. Then,

$$\begin{aligned}\|(X,Y)\|_{d_*(1,w^*)} &= \max\left\{\left|\frac{x+y}{1+w}\right|,\left|\frac{x-y}{1+w}\right|,\left(\frac{\left|\frac{x+y}{1+w}\right|+\left|\frac{x-y}{1+w}\right|}{1+w^*}\right)\right\}\\ &= \max\left\{\frac{|x|+|y|}{1+w},\frac{|x+y|+|x-y|}{2}\right\}\\ &= \max\left\{\frac{|x|+|y|}{1+w},\max\{|x|,|y|\}\right\}\\ &= \|(x,y)\|_{d_*(1,w)}.\end{aligned}$$

□

Lemma 5. *Let* $0<w<1, w^*=\frac{1-w}{1+w}$. *(a) Define* $\Phi\colon \mathcal{P}(^2d_*(1,w)^2)\to\mathcal{P}(^2d_*(1,w^*)^2)$ *by* $\Phi(P)=P\circ\phi^{-1}$, *where* ϕ *is the isometry in Lemma 4. Then,* Φ *is an isometric isomorphism. Therefore,* $P\in \mathrm{ext}B_{\mathcal{P}(^2d_*(1,w)^2)}$ *if and only if* $\Phi(P)\in\mathrm{ext}B_{\mathcal{P}(^2d_*(1,w^*)^2)}$.

(b) Define $\Psi\colon\mathcal{P}(^2d_*(1,w)^2)^*\to\mathcal{P}(^2d_*(1,w^*)^2)^*$ *by* $\Psi(f)(\Phi(P))=f(P)$ $(f\in\mathcal{P}(^2d_*(1,w)^2)^*, P\in\mathcal{P}(^2d_*(1,w)^2))$. *Then,* Ψ *is an isometric isomorphism. Therefore,* $P\in\mathrm{exp}B_{\mathcal{P}(^2d_*(1,w)^2)}$ *if and only if* $\Phi(P)\in\mathrm{exp}B_{\mathcal{P}(^2d_*(1,w^*)^2)}$.

If $f\in\mathcal{P}(^2d_*(1,w)^2)^*$, then $\Psi(f)(X^2)=(\frac{1+w^*}{2})^2(f(x^2)+f(y^2)+2f(xy))$, $\Psi(f)(Y^2)=(\frac{1+w^*}{2})^2(f(x^2)+f(y^2)-2f(xy))$, and $\Psi(f)(XY)=(\frac{1+w^*}{2})^2(f(x^2)-f(y^2))$. Note that if $P_t(x,y)=t(x^2-y^2)\pm2\sqrt{t(1-t)}xy$ $(\frac{1}{1+w^2}\le t\le1)$, then $\Phi(P_t)(X,Y)=\pm\frac{2\sqrt{t(1-t)}}{(1+w^*)^2}(X^2-Y^2)\pm\frac{4t}{(1+w^*)^2}XY$. From now on, the variable x and y will be used in the definition of polynomials in $\mathcal{P}(^2d_*(1,w)^2)$, whereas we use X and Y in the definition of polynomials in $\mathcal{P}(^2d_*(1,w^*)^2)$.

Theorem 6.

$$\begin{aligned}\mathrm{exp}B_{\mathcal{P}(^2d_*(1,w)^2)} = \mathrm{ext}B_{\mathcal{P}(^2d_*(1,w)^2)}\Big\backslash\Bigg\{&\pm\left[\frac{x^2-y^2\pm2wxy}{1+w^2}\right],\\ &\pm\left[\frac{1-w}{(1+w)(1+w^2)}(x^2-y^2)\pm\frac{2}{(1+w)^2}xy\right]\Bigg\}.\end{aligned}$$

Proof. We will use the following notations for the extreme points of $B_{\mathcal{P}(^2d_*(1,w)^2)}$:

$$P_t(x,y)=t(x^2-y^2)+2\sqrt{t(1-t)}xy\ \left(\frac{1}{1+w^2}\le t\le1\right),$$
$$\widetilde{P}_t(x,y)=t(x^2-y^2)-2\sqrt{t(1-t)}xy\ \left(\frac{1}{1+w^2}\le t\le1\right),$$
$$Q_s(x,y)=s(x^2-y^2)+\frac{2+2\sqrt{1-s^2(1+w)^4}}{(1+w)^2}xy\ \left(0\le s\le\frac{1-w}{(1+w)(1+w^2)}\right),$$

$$\widetilde{Q}_s(x,y)=s(x^2-y^2)-\frac{2+2\sqrt{1-s^2(1+w)^4}}{(1+w)^2}xy \quad \left(0\le s\le \frac{1-w}{(1+w)(1+w^2)}\right),$$
$$R_1(x,y)=x^2,$$
$$R_2(x,y)=y^2,$$
$$R_3(x,y)=\frac{1}{1+w^2}(x^2+y^2),$$
$$R_4(x,y)=\frac{1}{(1+w)^2}(x+y)^2,$$
$$R_5(x,y)=\frac{1}{(1+w)^2}(x-y)^2.$$

First, we will show that $R_1(x,y)\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Indeed, let $f\in \mathcal{P}(^2d_*(1,w)^2)^*$ be such that $f(x^2)=1,\ f(y^2)=\frac{w^2}{2},\ f(xy)=0$. By Theorem 3, $\|f\|=1$. We will show that f exposes R_1. Suppose that $Q(x,y)=ax^2+by^2+cxy\in\mathcal{P}(^2d_*(1,w)^2)$ such that $\|Q\|=1=f(Q)$. Then, $a+\frac{bw^2}{2}=1$. Since $\|(1,\frac{w\ \mathrm{sign}(c)}{\sqrt{2}})\|_{d_*}=1$ and $\|Q\|=1$, then $1+\frac{w|c|}{\sqrt{2}}=Q(1,\frac{w\ \mathrm{sign}(c)}{\sqrt{2}})\le 1$, from which we have $c=0$. For $0<t<w, \|(1,t)\|_{d_*}=1$ and $a+bt^2=Q(1,t)\le 1=a+\frac{bw^2}{2}$. As $t\to 0$, $b\ge 0$. As $t\to w$, $b\le 0$. Therefore, $b=0$ and $a=1$. So $Q(x,y)=R_1(x,y)$. Similarly, $-R_1(x,y),\ \pm R_2(x,y)\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. The latter shows also that $\pm X^2$ and $\pm Y^2$ are exposed polynomials in $B_{\mathcal{P}(^2d_*(1,w^*)^2)}$. Since $\frac{1}{(1+w)^2}(x+y)^2=\Phi^{-1}(X^2)$ and $\frac{1}{(1+w)^2}(x-y)^2=\Phi^{-1}(Y^2)$, by Lemma 5, $\pm R_4(x,y),\pm R_5(x,y)\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$.

We claim that $P_1(x,y)=x^2-y^2\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Indeed, $\|P_1\|=1$ by Theorem 1 and let $f\in\mathcal{P}(^2d_*(1,w)^2)^*$ be such that $f(x^2)=\frac{1}{2}, f(y^2)=-\frac{1}{2}, f(xy)=0$. We will show that f exposes P_1. Suppose that $Q(x,y)=ax^2+by^2+cxy\in\mathcal{P}(^2d_*(1,w)^2)$ such that $\|Q\|=1=f(Q)$. Since $1=f(Q)=\frac{1}{2}(a-b)$ and $|a|\le 1,\ |b|\le 1$, we have $a=1,\ b=-1$. We claim that $c=0$. Let $S(x,y)=x^2-y^2+|c|xy$. Since $\|S\|=1$ and $\|1,tw\|_{d_*}=1$ for every $0<t\le w$, $1-t^2w^2+|c|tw=S(1,tw)\le 1$ for every $0<t\le w$. As $t\to 0$, $c=0$. So $Q(x,y)=P_1(x,y)$. Similarly, $-P_1(x,y)\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Since $\frac{4}{(1+w)^2}xy=\Phi^{-1}(X^2-Y^2)$, by Lemma 5, $\pm\frac{4}{(1+w)^2}xy=\pm Q_0(x,y)\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Next, we will show that $R_3(x,y)\in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Let $f\in\mathcal{P}(^2d_*(1,w)^2)^*$ be such that $f(x^2)=\frac{1+w^2}{2}=f(y^2),\ f(xy)=0$. Obviously, $f(R_3)=1$ and by Theorem 3, $\|f\|=1$. We will show that f exposes R_3. Suppose that $Q(x,y)=ax^2+by^2+cxy\in\mathcal{P}(^2d_*(1,w)^2)$ is such that $\|Q\|=1=f(Q)$. Since $1=f(Q)$, $a+b=\frac{2}{1+w^2}$. Since $1<\frac{2}{1+w^2}<2$ and $|a|=|Q(1,0)|\le 1, |b|=|Q(0,1)|\le 1$, we have $a>0, b>0$. First, suppose that $a\ge b>0$. Put $a=\frac{1}{1+w^2}+t,\ b=\frac{1}{1+w^2}-t$ for $0\le t<\frac{1}{1+w^2}$. We claim that $t=0=c$. It follows that

$$1\ge |Q(1,\mathrm{sign}(c)w)|=1+t(1-w^2)+|c|w,$$

which shows that $t=0=c$. So $Q(x,y)=R_3(x,y)$. Suppose that $0<a\le b$. Let $a=\frac{1}{1+w^2}-l,\ b=\frac{1}{1+w^2}+l$ for $0\le l<\frac{1}{1+w^2}$. We also claim that $l=0=c$. It follows that

$$1 \geq |Q(\text{sign}(c)w, 1)| = 1 + l(1-w^2) + |c|w,$$

which shows that $l = 0 = c$. So $Q(x,y) = R_3(x,y)$. Similarly, $-R_3(x,y) \in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$.

Claim. $P_{\frac{1}{1+w^2}}(x,y) = \frac{1}{1+w^2}(x^2 - y^2 + 2wxy) \notin \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$.

Let $f \in \mathcal{P}(^2d_*(1,w)^2)^*$ be such that $1 = \|f\| = f(P_{\frac{1}{1+w^2}})$. Then, $(\alpha - \beta) + 2w\gamma = 1 + w^2$. By Theorem 1, $|\gamma| = |f(xy)| \leq \|f\|\|xy\| = \frac{(1+w)^2}{4}$. We claim that $0 \leq \alpha - \beta$. If $\alpha - \beta < 0$, then $\frac{1+w^2}{2w} < \gamma \leq \frac{(1+w)^2}{4}$, which is a contradiction. Since

$$\alpha - \beta = f(x^2 - y^2) \leq \|f\| = 1 = (\alpha - \beta) + 2w\gamma - w^2,$$

$\gamma \geq \frac{w}{2}$. Notice that $\frac{1}{1+w^2} \leq \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}}$ or $\frac{\alpha-\beta}{(1+w)^2\sqrt{(\alpha-\beta)^2+4\gamma^2}} \leq \frac{1-w}{(1+w)(1+w^2)}$. First, suppose that $\frac{1}{1+w^2} \leq \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}}$. Then, $\frac{1}{1+w^2} \leq \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}} \leq 1$. Define

$$g(x) = (\alpha - \beta)x + 2\sqrt{x(1-x)}\gamma \ \left(\frac{1}{1+w^2} \leq x \leq 1\right).$$

It follows that, by Theorem 3,

$$1 = \|f\| \geq g\left(\frac{1}{2} + \frac{\alpha - \beta}{2\sqrt{(\alpha-\beta)^2 + 4\gamma^2}}\right) = \frac{(\alpha - \beta) + \sqrt{(\alpha-\beta)^2 + 4\gamma^2}}{2},$$

which implies that, because of $\alpha - \beta = 1 + w^2 - 2w\gamma$,

$$2 \geq \alpha - \beta + \sqrt{(\alpha-\beta)^2 + 4\gamma^2} = 1 + w^2 - 2w\gamma + \sqrt{(1 + w^2 - 2w\gamma)^2 + 4\gamma^2},$$

which is equivalent to the inequality $1 - w^2 + 2w\gamma \geq \sqrt{(1+w^2-2w\gamma)^2 + 4\gamma^2}$, which reduces to the inequality $(\gamma - w)^2 \leq 0$, which implies that $\gamma = w$ and $\alpha - \beta = 1 - w^2$. Then, $1 = f(Q_{\frac{1-w}{(1+w)(1+w^2)}})$, which shows that f cannot expose $P_{\frac{1}{1+w^2}}$. Since f is arbitrary, we complete the proof of the claim if $\frac{1}{1+w^2} \leq \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}}$. Suppose that $\frac{\alpha-\beta}{(1+w)^2\sqrt{(\alpha-\beta)^2+4\gamma^2}} \leq \frac{1-w}{(1+w)(1+w^2)}$. Define

$$l(x) = (\alpha - \beta)x + \frac{2 + 2\sqrt{1 - (1+w)^4x^2}}{(1+w)^2}\gamma \ \left(0 \leq x \leq \frac{1-w}{(1+w)(1+w^2)}\right).$$

It follows that, by Theorem 3,

$$1 = \|f\| \geq l\left(\frac{\alpha - \beta}{(1+w)^2\sqrt{(\alpha-\beta)^2 + 4\gamma^2}}\right) = \frac{\sqrt{(\alpha-\beta)^2 + 4\gamma^2} + 2\gamma}{(1+w)^2},$$

which implies that, because of $\alpha - \beta = 1 + w^2 - 2w\gamma$,

$$(1+w)^2 - 2\gamma \geq \sqrt{(\alpha-\beta)^2 + 4\gamma^2} = \sqrt{(1+w^2-2w\gamma)^2 + 4\gamma^2},$$

which is equivalent to the inequality $(w^2 + 2w + 1 - 2\gamma)^2 - (w^2 + 1 - 2w\gamma)^2 \geq 4\gamma^2$, which reduces to the inequality $(\gamma - w)(w^2\gamma + w^2 + w + 1) = w^2\gamma^2 + (-w^3 + w^2 + w + 1)\gamma - w(w^2 + w + 1) \leq 0$, which implies that, since $w^2\gamma + w^2 + w + 1 >$

0, we have $\gamma \le w$. Note that the inequality $\frac{1}{1+w^2} \le \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}}$ is equivalent to the inequality $(\alpha-\beta)^2 + 4\gamma^2 \le (\frac{1+w^2}{1-w^2})^2(\alpha-\beta)^2$, which is also equivalent to the inequality $[w(1+w^2)+(1-3w^2)\gamma][-w(1+w^2)+(1+w^2)\gamma] \le 0$. Observe that $w(1+w^2)+(1-3w^2)\gamma > 0$ and $-w(1+w^2)+(1+w^2)\gamma \le 0$ because $\gamma \le w$. Hence, we have $\frac{1}{1+w^2} \le \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}}$.

The above argument of the case of $\frac{1}{1+w^2} \le \frac{1}{2} + \frac{\alpha-\beta}{2\sqrt{(\alpha-\beta)^2+4\gamma^2}}$ shows that f cannot expose $P_{\frac{1}{1+w^2}}$. Since f is arbitrary, we complete the proof of the claim if $\frac{\alpha-\beta}{(1+w)^2\sqrt{(\alpha-\beta)^2+4\gamma^2}} \le \frac{1-w}{(1+w)(1+w^2)}$.

Claim. $\widetilde{P}_{\frac{1}{1+w^2}}(x,y) \notin \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$.

Otherwise, there exists an $f \in \mathcal{P}(^2d_*(1,w)^2)^*$ with $1 = \|f\|$ which exposes $\widetilde{P}_{\frac{1}{1+w^2}}$. Let $\alpha = f(x^2), \beta = f(y^2), \gamma = f(xy)$. Then, $f(P_{\frac{1}{1+w^2}}) < 1$. Let $g \in \mathcal{P}(^2d_*(1,w)^2)^*$ be such that $g(x^2) = \alpha, g(y^2) = \beta, g(xy) = -\gamma$. Then, $g(P_{\frac{1}{1+w^2}}) = 1$. By Theorem 3, $\|g\| = 1$. Note that g exposes $P_{\frac{1}{1+w^2}}$. Indeed, let $Q(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ be such that $\|Q\| = 1 = g(Q)$. Let $\tilde{Q}(x,y) = Q(x,-y) = ax^2 + by^2 - cxy$. By Theorem 1, $\|\tilde{Q}\| = 1$ and $1 = g(Q) = f(\tilde{Q})$. Hence, $\tilde{Q} = \widetilde{P}_{\frac{1}{1+w^2}}$, which implies $Q = P_{\frac{1}{1+w^2}}$. Therefore, g exposes $P_{\frac{1}{1+w^2}}$, which is a contradiction because $P_{\frac{1}{1+w^2}} \notin \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Similarly, $-\widetilde{P}_{\frac{1}{1+w^2}}(x,y) \notin \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$. Since $\frac{1-w}{(1+w)(1+w^2)}(x^2-y^2) \pm \frac{2}{(1+w)^2}xy = \Phi^{-1}(\frac{1}{1+(w^*)^2}(X^2 - Y^2 \pm 2w^*XY))$, by Lemma 5, $\pm Q_{\frac{1-w}{(1+w)(1+w^2)}}(x,y), \pm\widetilde{Q}_{\frac{1-w}{(1+w)(1+w^2)}}(x,y) \notin \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$.

Claim. $P_t(x,y) \in \exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ *for* $\frac{1}{1+w^2} < t < 1$.

Let $\frac{1}{1+w^2} < t < 1$ be fixed. Let $f_t \in \mathcal{P}(^2d_*(1,w)^2)^*$ be such that $f_t(x^2) = 1 - \frac{1}{2t}, f_t(y^2) = -1 + \frac{1}{2t}$ and $f_t(xy) = \sqrt{\frac{1-t}{t}}$. Note that $|f_t(R_k)| < 1$ for $1 \le k \le 5$. We also have $f_t(P_t) = 1$ and $-1 < f_t(\widetilde{P}_l) < f_t(P_l) = l(2-\frac{1}{t}) + 2\sqrt{l(1-l)}\sqrt{\frac{1-t}{t}} < 1$ for $l \ne t, \frac{1}{1+w^2} \le l \le 1$. Hence,

$$(*)\ |f_t(R_k)| < 1, |f_t(P_l)| < 1, |f_t(\widetilde{P}_t)| < 1, |f_t(\widetilde{P}_l)| < 1 \text{ for } 1 \le k \le 5, \\ l \ne t, \frac{1}{1+w^2} \le l \le 1.$$

Define

$$\begin{aligned} h(s) &:= f_t(Q_s) \\ &= \left(2-\frac{1}{t}\right)s + \frac{2+2\sqrt{1-(1+w)^4s^2}}{(1+w)^2}\sqrt{\frac{1-t}{t}} \ \left(0 \le s \le \frac{1-w}{(1+w)(1+w^2)}\right). \end{aligned}$$

Then, since the only root of $h'(s) = 0$ in $[0, \frac{1-w}{(1+w)(1+w^2)}]$ is $\frac{2t-1}{(1+w)^2}$ and $h''(\frac{2t-1}{(1+w)^2}) = 0$, we have

$$\sup_{0\le s\le \frac{1-w}{(1+w)(1+w^2)}} h(s) = h\left(\frac{2t-1}{(1+w)^2}\right) = \frac{1+2\sqrt{t(1-t)}}{(1+w)^2 t} < 1.$$

Hence, $-1 < f_t(\widetilde{Q}_s) < f_t(Q_s) < 1$ for $0 \le s \le \frac{1-w}{(1+w)(1+w^2)}$. Hence,

$$(**)\ |f_t(Q_s)| < 1 \quad \text{and} \quad |f_t(\widetilde{Q}_s)| < 1 \quad \text{for } 0 \le s \le \frac{1-w}{(1+w)(1+w^2)}.$$

By Theorem 3, $\|f_t\| = 1$. We will show that f_t exposes P_t. Let $Q(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2d_*(1,w)^2)$ such that $1 = \|Q\| = f_t(Q)$. We will show that $Q = P_t$. Since $\mathcal{P}(^2d_*(1,w^*)^2)$ is a finite dimensional Banach space with dimension 3, by the Krein–Milman theorem, $B_{\mathcal{P}(^2d_*(1,w^*)^2)}$ is the closed convex hull of $\mathrm{ext}B_{\mathcal{P}(^2d_*(1,w^*)^2)}$. Then,

$$\begin{aligned} Q(x,y) = &\sum_{1\le k\le 5} \lambda_k R_k(x,y) + \sum_{j=1}^{\infty} \beta_j Q_{s_j}(x,y) + \sum_{n=1}^{\infty} \gamma_n \widetilde{Q}_{s'_n}(x,y) \\ &+ \sum_{m=1}^{\infty} \delta_m P_{t_m}(x,y) + \sum_{l=1}^{\infty} \epsilon_l \widetilde{P}_{t'_l}(x,y) \end{aligned}$$

for some $\lambda_k, \beta_j, \gamma_n, \delta_m, \epsilon_l \in \mathbb{R}$ with

$$(***)\ \sum_{1\le k\le 5} |\lambda_k| + \sum_{j=1}^{\infty} |\beta_j| + \sum_{n=1}^{\infty} |\gamma_n| + \sum_{m=1}^{\infty} |\delta_m| + \sum_{l=1}^{\infty} |\epsilon_l| \le 1$$

and some $0 \le s_j, s'_n \le \frac{1-w}{(1+w)(1+w^2)}$ and $\frac{1}{1+w^2} \le t_m, t'_l \le 1$ for every $j,n,m,l \in \mathbb{N}$.

Claim. $\lambda_k = \beta_j = \gamma_n = 0$, *for every* $1 \le k \le 5, j, n \in \mathbb{N}$.

Assume that $\lambda_{k_0} \ne 0$ for some $1 \le k_0 \le 5$. It follows that

$$\begin{aligned} 1 = f_t(Q) = &\sum_{1\le k\le 5} \lambda_k f_t(R_k) + \sum_{j=1}^{\infty} \beta_j f_t(Q_{s_j}) + \sum_{n=1}^{\infty} \gamma_n f_t(\widetilde{Q}_{s'_n}) \\ &+ \sum_{m=1}^{\infty} \delta_m f_t(P_{t_m}) + \sum_{l=1}^{\infty} \epsilon_l f_t(\widetilde{P}_{t'_l}) \\ \le &|\lambda_{k_0}|\ |f_t(R_{k_0})| + \sum_{1\le k\ne k_0\le 5} |\lambda_k||f_t(R_k)| + \sum_{j=1}^{\infty} |\beta_j||f_t(Q_{s_j})| + \sum_{n=1}^{\infty} |\gamma_n||f_t(\widetilde{Q}_{s'_n})| \\ &+ \sum_{m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + \sum_{l=1}^{\infty} |\epsilon_l||f_t(\widetilde{P}_{t'_l})| \\ < &|\lambda_{k_0}| + \sum_{1\le k\ne k_0\le 5} |\lambda_k||f_t(R_k)| + \sum_{j=1}^{\infty} |\beta_j||f_t(Q_{s_j})| + \sum_{n=1}^{\infty} |\gamma_n||f_t(\widetilde{Q}_{s'_n})| \\ &+ \sum_{m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + \sum_{l=1}^{\infty} |\epsilon_l||f_t(\widetilde{P}_{t'_l})| \text{ (by (*))} \end{aligned}$$

$$\leq |\lambda_{k_0}| + \sum_{1\leq k\neq k_0\leq 5} |\lambda_k| + \sum_{j=1}^{\infty} |\beta_j| + \sum_{n=1}^{\infty} |\gamma_n| + \sum_{m=1}^{\infty} |\delta_m| + \sum_{l=1}^{\infty} |\epsilon_l| \text{ (by (*))}$$
$$\leq 1 \text{ (by (***))},$$

which is impossible. Therefore, $\lambda_k = 0$, for every $1 \leq k \leq 5$.

Assume that $\beta_{j_0} \neq 0$ for some $j_0 \in \mathbb{N}$. Using a similar argument as above, we have

$$\begin{aligned}
1 = f_t(Q) &= \sum_{j=1}^{\infty} \beta_j f_t(Q_{s_j}) + \sum_{n=1}^{\infty} \gamma_n f_t(\widetilde{Q}_{s'_n}) \\
&\quad + \sum_{m=1}^{\infty} \delta_m f_t(P_{t_m}) + \sum_{l=1}^{\infty} \epsilon_l f_t(\widetilde{P}_{t'_l}) \\
&\leq |\beta_{j_0}|\, |f_t(Q_{s_{j_0}})| + \sum_{j\neq j_0, j=1}^{\infty} |\beta_j||f_t(Q_{s_j})| + \sum_{n=1}^{\infty} |\gamma_n||f_t(\widetilde{Q}_{s'_n})| \\
&\quad + \sum_{m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + \sum_{l=1}^{\infty} |\epsilon_l||f_t(\widetilde{P}_{t'_l})| \\
&< |\beta_{j_0}| + \sum_{j\neq j_0, j=1}^{\infty} |\beta_j||f_t(Q_{s_j})| + \sum_{n=1}^{\infty} |\gamma_n||f_t(\widetilde{Q}_{s'_n})| \\
&\quad + \sum_{m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + \sum_{l=1}^{\infty} |\epsilon_l||f_t(\widetilde{P}_{t'_l})| \text{ (by (**))} \\
&\leq \sum_{j=1}^{\infty} |\beta_j| + \sum_{n=1}^{\infty} |\gamma_n| + \sum_{m=1}^{\infty} |\delta_m| + \sum_{l=1}^{\infty} |\epsilon_l| \\
&\leq 1,
\end{aligned}$$

which is impossible. Therefore, $\beta_j = 0$, for every $j \in \mathbb{N}$. Using a similar argument as above, we have $\gamma_n = 0$, for every $n \in \mathbb{N}$. Therefore,

$$Q(x, y) = \sum_{m=1}^{\infty} \delta_m P_{t_m}(x, y) + \sum_{l=1}^{\infty} \epsilon_l \widetilde{P}_{t'_l}(x, y).$$

We claim that for every $l \in \mathbb{N}$, then $\epsilon_l = 0$. Assume that $\epsilon_{l_0} \neq 0$ for some $l_0 \in \mathbb{N}$. Then,

$$\begin{aligned}
1 = f_t(Q) &= \sum_{m=1}^{\infty} \delta_m f_t(P_{t_m}) + \sum_{l=1}^{\infty} \epsilon_l f_t(\widetilde{P}_{t'_l}) \\
&\leq \sum_{m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + |\epsilon_{l_0}||f_t(\widetilde{P}_{t'_{l_0}})| + \sum_{l\neq l_0, l=1}^{\infty} |\epsilon_l||f_t(\widetilde{P}_{t'_l})| \\
&< \sum_{m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + |\epsilon_{l_0}| + \sum_{l\neq l_0, l=1}^{\infty} |\epsilon_l||f_t(\widetilde{P}_{t'_l})| \text{ (by (*)}
\end{aligned}$$

$$\leq \sum_{m=1}^{\infty} |\delta_m| + \sum_{l=1}^{\infty} |\epsilon_l|$$
$$\leq 1,$$

which is impossible. Therefore, $\epsilon_l = 0$ for every $l \in \mathbb{N}$. So

$$Q(x,y) = \sum_{m=1}^{\infty} \delta_m P_{t_m}(x,y).$$

We will show that if $t_m \neq t$ for some $m \in \mathbb{N}$, then $\delta_m = 0$. Suppose that $t_{m_0} \neq t$ for some $m_0 \in \mathbb{N}$. Assume that $\delta_{m_0} \neq 0$.

$$\begin{aligned}
1 = f_t(Q) &= \sum_{m=1}^{\infty} \delta_m f_t(P_{t_m}) \\
&\leq |\delta_{m_0}||f_t(P_{t_{m_0}})| + \sum_{m \neq m_0, m=1}^{\infty} |\delta_m||f_t(P_{t_m})| \\
&< |\delta_{m_0}| + \sum_{m \neq m_0, m=1}^{\infty} |\delta_m||f_t(P_{t_m})| + \text{ (by (*))} \\
&\leq \sum_{m=1}^{\infty} |\delta_m| \\
&\leq 1,
\end{aligned}$$

which is impossible. Hence, $\delta_{m_0} = 0$. Therefore,

$$Q(x,y) = \left(\sum_{m=1}^{\infty} \delta_m \right) P_t(x,y) = P_t(x,y),$$

from which $P_t(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ for $\frac{1}{1+w^2} < t < 1$. Similarly, $\pm \widetilde{P}_t(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ for $\frac{1}{1+w^2} < t < 1$.

Claim. $Q_s(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ *for* $0 < s < \frac{1-w}{(1+w)(1+w^2)}$.

By Lemma 5, it is enough to show that $\Phi(Q_s) \in \exp B_{\mathcal{P}(^2 d_*(1,w^*)^2)}$. It follows that

$$\begin{aligned}
\Phi(Q_s)(X,Y) &= Q_s \circ \phi^{-1}(X,Y)) \\
&= \left(s(x^2 - y^2) + \frac{2 + 2\sqrt{1 - s^2(1+w)^4}}{(1+w)^2} xy \right) \circ \phi^{-1}(X,Y) \\
&= s \left(\left(\frac{1+w}{2} \right)^2 (X+Y)^2 - \left(\frac{1+w}{2} \right)^2 (X-Y)^2 \right) \\
&\quad + \frac{2 + 2\sqrt{1 - s^2(1+w)^4}}{(1+w)^2} \left(\frac{1+w}{2} \right)^2 (X^2 - Y^2) \\
&= \frac{1 + \sqrt{1 - s^2(1+w)^4}}{2} (X^2 - Y^2) + (1+w)^2 sXY.
\end{aligned}$$

Let $t = \frac{1+\sqrt{1-s^2(1+w)^4}}{2}$. Then, $\frac{1}{1+(w^*)^2} < t < 1$ and $\Phi(Q_s)(X,Y) = P_t(X,Y) \in \exp B_{\mathcal{P}(^2 d_*(1,w^*)^2)}$. Similarly, $-Q_s(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ for $0 < s < \frac{1-w}{(1+w)(1+w^2)}$.

Claim. $\widetilde{Q}_s(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ *for* $0 < s < \frac{1-w}{(1+w)(1+w^2)}$.

Since $Q_s(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$, there exists an $f \in \mathcal{P}(^2 d_*(1,w)^2)^*$ with $1 = \|f\|$ which exposes Q_s. Let $\alpha = f(x^2), \beta = f(y^2), \gamma = f(xy)$. Let $g \in \mathcal{P}(^2 d_*(1,w)^2)^*$ be such that $g(x^2) = \alpha, g(y^2) = \beta, g(xy) = -\gamma$. Then, $g(\widetilde{Q}_s) = 1$. By Theorem 3, $\|g\| = 1$. Note that g exposes $\widetilde{Q}_s$. Indeed, let $Q(x,y) = ax^2 + by^2 + cxy \in \mathcal{P}(^2 d_*(1,w)^2)$ be such that $\|Q\| = 1 = g(Q)$. Let $\tilde{Q}(x,y) = Q(x,-y) = ax^2 + by^2 - cxy$. By Theorem 1, $\|\tilde{Q}\| = 1$ and $1 = g(Q) = f(\tilde{Q})$. Hence, $\tilde{Q} = \widetilde{Q}_s$, which shows that g exposes $\widetilde{Q}_s$. Similarly, $-\widetilde{Q}_s(x,y) \in \exp B_{\mathcal{P}(^2 d_*(1,w)^2)}$ for $0 < s < \frac{1-w}{(1+w)(1+w^2)}$. Therefore, we complete the proof of Theorem 6. □

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